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Potts model on a Cayley tree and logistic equation

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Abstract. The q-state Potts model on a Cayley tree can be solved by a recursion formula depending on the properties at the surface. The model on a Bethe lattice is obtained by extrapolating the interior of a Cayley tree sufficiently far from the surface in order to have a stable fixed point of the recursion. For $q > 2$ we find a second-order transition of percolation type at the Bethe–Peierls temperature and a first-order transition at a higher temperature. For coordination number $z = 3$ the recursion extrapolated to $q = 1$ is identical to the logistic equation. The Feigenbaum route to chaos appears for antiferromagnetic coupling of the Potts model. The first period doubling corresponds to a multicritical point in the phase diagram of the Potts model.

1. Introduction

The q-state Potts model [1,2] ought to be solvable on a Cayley tree with coordination number z, since loop expansion of zero order [3] or mean field should be exact on an infinite-dimensional lattice. However, one has to distinguish [4, 5] between a large finite Cayley tree and the interior of an infinitely extended tree, also called a Bethe lattice. Extrapolation of the former is nontrivial, since the surface of a Cayley tree cannot be neglected. One interesting aspect of this model is that it can serve as a testing ground for properties of phase transitions. Zero-order loop expansion or Bethe–Peierls approximation [3] yield a second-order phase transition [6–10] at the inverse Bethe–Peierls temperature $K_c = \ln(1 + q/(z - 2))$ for all q and $z > 2$. In contrast, one obtains at a temperature above $\frac{1}{K_c}$, by applying Landau theory [11, 12], a first-order transition for $q > 2$. The claim of [13, 14] for unusual transitions in the Ising case $q = 2$ contributed additional confusion. A partial answer to this contradiction has been given by Peruggi *et al* [15], who calculated the free energy per site recursively on a Bethe lattice. They found for $q > 2$ a first-order transition above the Bethe–Peierls temperature. K_c itself turned out to be the endpoint of spinodal curves due to metastable states (see also [16]). The unusual transitions mentioned above are due to the dominance of surface sites. Those models and possible generalizations [17–21] may be interesting, but are not related to the Bethe lattice. The work of Peruggi *et al* [15] leaves open the question of how to obtain these results from the extrapolation of a Cayley tree. In a first attempt Gujrati [22] derived a recursion formalism valid for a Cayley tree of arbitrary magnitude. Similar methods were applied later in [23]. Using only fixed points the results of [15] have been confirmed, but the stability of the recursion has not been discussed. One of the aims of this paper is to use this stability to distinguish different phases. Another interesting point of the Potts model lies in considering its thermodynamic functions as analytic functions of the number q of states. Values $q \leqslant 1$ yield the connection of various models to statistical physics [1]. Extrapolation to $q = 1$ in the zero-field case ($L = 0$) for ferromagnetic coupling ($K > 0$) is usually interpreted as bond percolation [24]. The

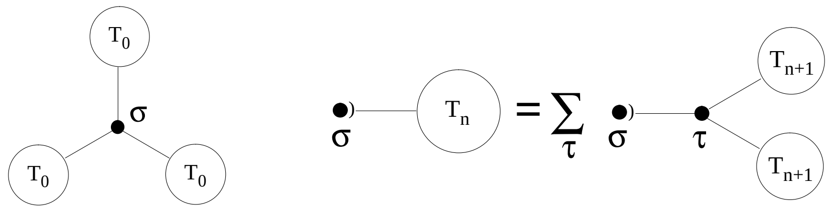


Figure 1. The left part gives the decomposition of the Cayley tree into z branches connected to an origin with spin σ and shell index $n = 0$. The right part illustrates the recursion relation to compute a branch starting at shell n from those starting at $n + 1$. Both figures are examples of the coordination number $z = 3$.

question remains, however, of how the $q = 1$ limit is to be interpreted for $L \neq 0$, especially in the case of antiferromagnetic coupling $K < 0$.

This paper is organized in the following way. In section 2 we sketch the recursion formalism of Gujrati [22]. We also derive the correlation function generalizing the matrix method of [15] and its related observables. Section 3 contains the general discussion of stability and its relation to the free energy of [15] on the Bethe lattice. In section 4 we apply our stability criterion to the zero-external-field phase diagram for $q \geqslant 2$. We argue that both first- and second-order transitions will appear for $q > 2$. In section 5 we discuss the limit $q = 1$. The recursion relation connected to the magnetization becomes identical to the logistic equation. Consequences of the transition to chaotic behaviour of the latter for the $q = 1$ Potts model are discussed. In section 6 we investigate the relation to a generalized percolation model valid also for $K < 0$ and $L \neq 0$. In the conclusion (section 7) we summarize our results.

2. General recursion formula

We consider the q-state Potts model on a Cayley tree of coordination number z. It can be interpreted as z branches with R shells connected to an origin with shell number $n = 0$. At each site there sits a spin $\sigma_i = 1, 2, \ldots, q$ interacting with an external field which distinguishes a fixed spin value $\bar{\sigma}$. Two nearest-neighbour spins σ_i and σ_j contribute a term $K\delta_{\sigma_i,\sigma_j}$ to the Hamiltonian $-\beta H$, i.e.

$$-\beta H = K \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} + \sum_i L_i \delta_{\sigma_i,\bar{\sigma}}. \tag{1}$$

The first sum in equation (1) extends over all nearest-neighbouring pairs $\langle ij \rangle$ of the lattice. The external field L_i appearing in the second sum is made up of two parts: $L_i = L + (-1)^{n(i)} L_s$, where L denotes a constant magnetic field and the second term stands for a staggered field, its sign alternating from one shell to the next. The partition sum can be written [22] as a product of partition sums $T_0(\sigma)$ (see figure 1) of branches of length R summed over the spin σ at the origin

$$Z_R(K, L) = \sum_\sigma \mathrm{e}^{L\delta_{\sigma,\bar{\sigma}}} (T_0(\sigma))^z. \tag{2}$$

The factors T_0 for the branches can be calculated recursively (see [22] and earlier work quoted therein). One can parametrize the partition sum $T_n(\sigma)$ for a branch starting at shell n by

$$T_n(\sigma) = a_n(\delta_{\sigma,\bar{\sigma}} + x_n(1 - \delta_{\sigma,\bar{\sigma}})). \tag{3}$$

For the coefficients x_n and a_n one finds (see figure 1) a recursion formula:

$$x_n = \frac{\mathrm{e}^L + (\mathrm{e}^K + q - 2)(x_{n+1})^{z-1}}{A_1(x_{n+1})} \tag{4}$$

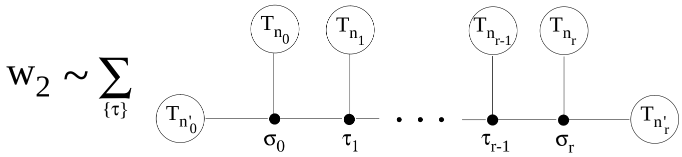


Figure 2. Graphical representation of Boltzmann factors. The correlations $w_2(\sigma_0, \sigma_r, r)$ are obtained by summing over all spins except the external spins σ_0 and σ_r. The one-spin probability $w_1(\sigma_0)$ is found by setting $r = 0$.

$$a_n = (a_{n+1})^{z-1} A_1(x_{n+1}) \tag{5}$$

with the abbreviation

$$A_1(x) = \mathrm{e}^{K+L} + (q-1)x^{z-1}. \tag{6}$$

Equations (4) and (5) allow the recursive compution of T_n for $n < n_0$ given the values x_{n_0} and a_{n_0}. The solution of (3) and (4) with arbitrary values at the surface $n_0 = R$ of the branches is technically difficult. If the map (4) has a stable fixed point or a stable orbit of period τ, we can choose $1 \ll n_0 \ll R$. For large enough n_0 one can replace x_n with $n < n_0$ by the fixed point and a_n can be found as a function of a_{n_0}. Knowing T we can determine Z_R as a function of K, L and a_{n_0}. The latter cannot be eliminated, since (5) is always unstable, i.e. sensitive to the initial values. However, knowledge of a_{n_0} is not needed, if we want to calculate the Boltzmann distribution or correlation function $w_2(\sigma_0, \sigma_r, r)$ for two spins residing at site 0 and a second site at distance r.

Two sites on a Cayley tree can be connected by a line, as illustrated in figure 2. The constants a_n contained in the branches adjacent to the connecting line cancel in w_2 due to the normalization condition $\sum_{\sigma,\sigma'} w_2(\sigma, \sigma') = 1$. Assuming both sites are inside the shell n_0 we can set x_n in all T along the line equal to the fixed-point value. Then w_2 is translation invariant and independent of the size n_0 of the subsystem. Its behaviour is the same on an infinitely extended tree or Cayley tree. Putting the distance in w_2 to $r = 0$ we obtain the Boltzmann distribution for spin σ

$$\begin{aligned} w_1(\sigma) &= \frac{1}{Z_R} \mathrm{e}^{L\delta_{\sigma,\bar\sigma}} (T_R(\sigma))^z \\ &= \frac{\mathrm{e}^L \delta_{\sigma,\bar\sigma} + x^z(1-\delta_{\sigma,\bar\sigma})}{A_2(x)} \end{aligned} \tag{7}$$

with the function

$$A_2(x) = \mathrm{e}^L + (q-1)x^z. \tag{8}$$

An observable related to the Bolzmann distribution (7) is the magnetization

$$m(x) = \frac{q}{q-1}\left(w_1(\bar\sigma) - \frac{1}{q}\right). \tag{9}$$

Inserting the form (7) we find

$$m(x) = \frac{\mathrm{e}^L - x^z}{A_2(x)}. \tag{10}$$

The q-dependent normalization in (9) is chosen conventionally [1] in order that the cases $m = 0$ or $m = 1$ correspond to a disordered ($x = 1$) resp. a fully ordered spin state on the lattice ($x = 0$). Moreover, (10) can be extrapolated to $q = 1$ with a nontrivial result.

The two-point function w_2 for a fixed point of (4) can be obtained by the matrix method of [15]. This formalism can be generalized to the case of an orbit $x_\pm$ of period $\tau = 2$. The fixed-point case can be recovered by setting $x_+ = x_- = x$. The result of a rather tedious and lengthy calculation of w_2 is

$$w_2(\sigma_0, \sigma_r, r) = w_1(\sigma_0)w_1(\sigma_r) + \frac{(1 - m(x_0))x_r}{qx_0}[S_\perp(\sigma_0, \sigma_r)\Gamma(\epsilon_\perp, r) \\ +(1 + (q-1)m(x_r))S_\parallel(\sigma_0, \sigma_r)\Gamma(\epsilon_\parallel, r)] \tag{11}$$

with

$$\Gamma(\epsilon, r) = \begin{cases} (\epsilon(x_+)\epsilon(x_-))^{r/2} & r \text{ even} \\ (\epsilon(x_+)\epsilon(x_-))^{(r-1)/2}\epsilon(x_r) & r \text{ odd} \end{cases} \tag{12}$$

and the spin factors

$$S_\parallel(\sigma_0, \sigma_r) = \frac{1}{q(q-1)}(q\delta_{\sigma_0,\bar\sigma} - 1)(q\delta_{\sigma_r,\bar\sigma} - 1) \tag{13}$$

$$S_\perp(\sigma_0, \sigma_r) = \frac{1}{q-1}(1 - \delta_{\sigma_0,\bar\sigma})(1 - \delta_{\sigma_r,\bar\sigma})((q-1)\delta_{\sigma_0,\sigma_r} - 1). \tag{14}$$

The form of Γ shows that the correlations decay exponentially with the decay constants

$$\epsilon_\perp(x) = x^{z-2}(\mathrm{e}^K - 1)/A_1(x) \tag{15}$$

and

$$\epsilon_\parallel(x) = \epsilon_\perp(x)\left(1 + \frac{(q-1)(1-x)}{\mathrm{e}^K - 1}\right). \tag{16}$$

The same formulae (11)–(16) hold also for a fixed-point solution with $x_+ = x_-$. The first term in (11) corresponds to the unconnected part, the second to correlations if both spins have no $\bar\sigma$-component only relevant for $q > 2$ and the third contributes to correlations, if both points have spins equal to $\bar\sigma$. The connected two-point function in the same normalization as the magnetization (9) is given by

$$w_2^c = \frac{q}{q-1}(w_2 - w_1(\sigma_0)w_1(\sigma_r)). \tag{17}$$

For phase transitions correlations are important where at least one of the spins is $\bar\sigma$. The other spin may be equal or not equal to $\bar\sigma$. These two cases we index by $\lambda(r) = \pm 1$. For those correlations the spin factors reduce to $S_\parallel = \lambda(r)(q-1)/q$ and $S_\perp = 0$. Inserting w_2 from (11) into (17) we find

$$w_2^c(\lambda, r) = \frac{\lambda x_0}{qx_r}(1 - m(x_0))(1 + (q-1)m(x_r))\Gamma(\epsilon_\parallel, r). \tag{18}$$

The choice $\lambda(r) = 1$ corresponds to ferromagnetic and $\lambda(r) = (-)^r$ to antiferromagnetic correlations. As m, w_2^c from (18) can also be extrapolated to $q = 1$ with a nontrivial result. Since for a period 2 orbit solution of (4) adjacent shells have magnetization $m(x_\pm)$, we define the average (m_0) and and the staggered (m_s) magnetizations by

$$m_{0,s} = \tfrac{1}{2}(m(x_+) \pm m(x_-)). \tag{19}$$

An observable related to (18) is the susceptibility. Even at $L_s = 0$ there exist four possibilities. Since susceptibilities are sums over w_2^c having only one decay parameter $\epsilon_\parallel$, there are only two independent possibilities for which we take the average susceptibility

$$\chi_0 = \sum_r w_2^c(1, |r|) \tag{20}$$

and the staggered susceptibility

$$\chi_s = \sum_r w_2^c((-1)^{|r|}, |r|) \tag{21}$$

where the sum comprises all sites and $|r|$ denotes the distance of the two points. χ_s differs from the normal susceptibility χ_0, that at points j with odd $|j|$ the antiferromagnetic correlation has to be taken. It is easy to check that $\chi_{0,s}$ satisfy the relation $\chi_{0,s} = \partial m(x_+)/\partial L_{0,s}|_{L_{0,s}=0}$, where L_s denotes a staggered external field. The recursion relation analogous to (4) in the presence of L_s can be derived, and by expansion around $L_s = 0$ the derivatives with respect to L_s at $L_s = 0$ can be computed. The reason for considering χ_s is the behaviour at critical points. Since the number of points in distance r will increase with $(z-1)^r$, the susceptibility (20), (21) will diverge if $|\epsilon_+\epsilon_-| = (z-1)^{-2}$ provided all $\Gamma_\| > 0$. If $\epsilon_\|$ becomes negative (for $K < 0$) χ_s is the sum of positive quantities and will exhibit a divergence, whereas χ_0 remains finite. We notice that by choosing an appropriate $\lambda(r)$ we can always achieve $w_2^c > 0$ for $r > 0$.

3. Stability and free energy

Any observable computed from the functions $w_{1,2}$ in a region $n < n_0$ is the same as for an infinitely extended Bethe lattice. The recursion formula (4) must have a stable orbit of length τ $x_t^*(t = 1, \ldots, \tau)$ for obtaining these observables independent of the outside region $n > n_0$. Stability occurs if

$$D = \left| \prod_{t=1}^{\tau} \frac{\partial x_t^*}{\partial x_{t+1}^*} \right| < 1 \tag{22}$$

holds [25]. In the following we restrict for simplicity the discussion to the case of a fixed point in which (22) reads as

$$D(K, L) = \left| \frac{\partial x_n}{\partial x_{n+1}} \bigg|_{x_{n+1}=x} \right| < 1. \tag{23}$$

In all previous publications this stability condition has been simply ignored. A problem arises if the recursion formula (4) allows more than one solution with $D < 1$. Adopting our method the true boundary conditions at the surface of the Cayley tree at $n = R$ decide which of the possible states are present at $n = n_0$. States with different x may have different basins of attraction in the space of the boudary values at $n = R$. Therefore, phase transitions occur either at $D(K, L) = 1$ or at values of K, L where the solutions of (4) are no longer real. This is, in general, at variance with the standard method adopted from finite-dimensional lattices (an example is given in section 4). One has to determine from Z the free energy per point $f_B(m)$ as a function of m describing the system on the Bethe lattice. The zero-field solutions correspond to minima of $f_B(m)$. Only the global minimum is stable, all others are metastable. Criticality is given by $f_B(m(x_1)) = f_B(m(x_2))$. Those obtained by $D(K, L) = 1$ or Im $(x) = 0$ would be interpreted as spinodal points. In the following we show that this method can be questioned due to the dependence on the boundary conditions. The method to calculate $f_B(m)$ has been given in [23]. For the logarithm of the partition function per point ω_B we have two conditions:

$$\frac{\partial \omega_B}{\partial L} = w_1(\bar{\sigma}) \tag{24}$$

$$\frac{\partial \omega_B}{\partial K} = \frac{z}{2} \sum_\sigma w_2(\sigma, \sigma, 1). \tag{25}$$

The factor $z/2$ in front of the energy per point (25) accounts for the ratio of the number of bonds to sites on a Bethe lattice. As in the $q = 2$ case [5] the differential equations (24) and (25) can be integrated with the result

$$\omega_B = \frac{z}{2} \ln \frac{A_1(x)}{A_2(x)} + \ln A_2(x). \tag{26}$$

The functions $A_{1,2}(x)$ are given in (6) and (8). To obtain ω_B within our recursion formalism we solve (5) for $n < n_0 \gg 1$

$$\ln a_n = (z-1)^{n_0-n} \ln(a_{n_0} A_1^{1/(z-2)}) \tag{27}$$

and obtain for $\omega = (z/N(n_0)) \ln a_0$

$$\omega = (z-2) \ln a_{n_0} + \ln A_1. \tag{28}$$

With the nontrivial value $a_{n_0}^2 = A_1/A_2$ we achieve equality of ω and ω_B. By a Legendre transformation [22] one obtains $f_B(K, m)$ from $\omega_B(K, L)$. A stability criterion based on f_B assumes that L inside the shell n_0 can be varied freely. This implies also a change of a_{n_0}. Since a_{n_0} is connected to the true boundary values at $n = R$ by

$$\ln a_{n_0} = \frac{N(R)}{N(n_0)} \left(\ln a_R + \sum_{\nu=n_0+1}^{R} (z-1)^{\nu-1-R} \ln A_1(x_\nu) \right) \tag{29}$$

we encounter a fine-tuning problem for $N(R) \gg N(n_0)$. Small changes in a_R or x_R will change a_{n_0} by large amounts. Therefore stability of the whole Cayley tree has to be considered. Starting from arbitrary values a_R and x_R at the surface a stable fixed point guarantees stable observables (24) and (25). This means that f_B describes the observables in thermal equilibrium, but should not be used to rule out possible stable fixed points of recursion (4). In [22] a proposal to compute f_B independently of a_R and x_R has been given without proof. The instability with respect to surface will make it difficult to prove this prescription.

4. Phases for $q \geqslant 2$ at zero field

If the external field L vanishes, the recursion relation (4) can be solved at least qualitatively. The fixed points are the zeros of the following function:

$$R(x) = \frac{1 + (e^K + q - 2)x^{z-1}}{e^K + (q-1)x^{z-1}} - x \tag{30}$$

which is depicted in the case $z = q = 3$ in figure 3 for various values of K. $R(x)$ behaves similarly for all $z, q > 2$. A fixed point is stable according to (23), if R has a negative slope no smaller than -2. The function (30) has always the zero $x_1 = 1$, which corresponds to the disordered phase. The fixed point $x_1 = 1$ is stable in the range

$$\ln\left(1 - \frac{q}{z}\right) = K_c' < K < K_c = \ln\left(1 + \frac{q}{z-2}\right). \tag{31}$$

The upper limit K_c agrees with the Bethe–Peierls temperature obtainable by loop expansion [3]. At negative and small positive values of K only the solution $x_1 = 1$ exists. Above a critical K_c'' a pair of two further solutions $x_{2,3}$ appear (see figure 3). The solution x_2 with $x_2 < 1$ satisfies $D(x_2) < 1$ and is therefore stable for all $K > K_c''$. The other solution x_3 with $x_3 > 1$ has negative magnetization and is stable above K_c, where the disordered solution becomes unstable. Above K_c'' two possible states always exist. The boundary values decide which of the two is adopted on the lattice. In the case of $z = 3$ the value of K_c'' is given by

$$K_c'' = \ln\left(1 + 2\sqrt{q-1}\right) < K_c. \tag{32}$$

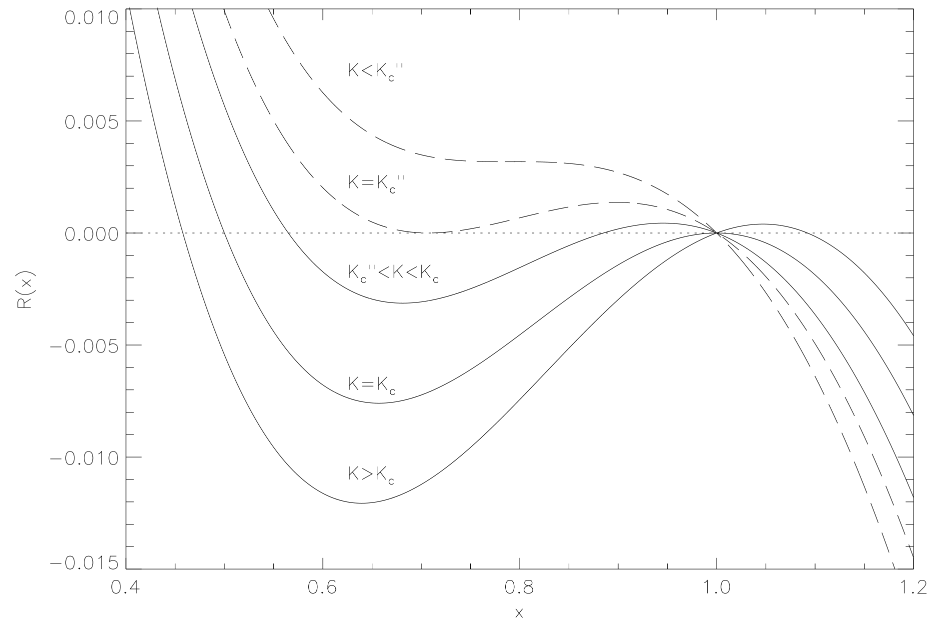


Figure 3. Function $R(x)$ for $z = 3$ given by (30) as function of x for $L = 0$ and various K. Solid curves are for $K > K_c''$, dashed curves for $K \leqslant K_c''$. Stable fixed points correspond to zeros with negative slope.

The transition at K_c'' is of first order since $m(x_2(K_c'')) \neq 0$. Below the Bethe–Peierls temperature $1/K_c$ a negatively magnetized phase $m(x_3) < 0$ replaces the disordered phase. This transition $x_1 \leftrightarrow x_3$ is of second order since $x_3(K_c) = 1$ holds. Expanding the fixed point $x(K, L)$ of (4) around the values $K = K_c$ and $L = 0$ up to order $(K - K_c)$, L we find for the magnetization as a function of K

$$m(K)|_{L=0} = -\frac{2(z-2)q(K-K_c)}{(q-2)(z-1)} \tag{33}$$

and as a function of L

$$m(L)|_{K=K_c} = -\frac{q}{z}\sqrt{\frac{2qL}{(q-2)(z-2)(z-1)}}. \tag{34}$$

Taking the derivative of m with respect to L we get the zero-field susceptibility near K_c

$$\chi(K) = \left.\frac{\partial m}{\partial L}\right|_{L=0} = \frac{1}{z}\left(\frac{q}{z-2}\right)^2 |K - K_c|^{-1}. \tag{35}$$

From equations (33)–(35) we read off the critical indices $\beta = 1$, $\delta = 2$ and $\gamma = 1$. Note that L has to be positive near K_c, otherwise no fixed point will exist. In the antiferromagnetic case $K < 0$ there exists no positive fixed point besides $x_1 = 1$. If we decrease K below K_c' the iterated recursion formula exhibits a stable orbit $x_\pm$ of length $\tau = 2$ corresponding to an antiferromagnetic ordering, since the magnetization alternates from shell to shell. This can occur only for $2 \leqslant q < z$. The critical point K_c' has been found already in [15].

The Ising case $q = 2$ is exceptional due to its global symmetry, which states that any two fixed points $x_\pm$ satisfy $x_+x_- = 1$. This implies the relation $K_c'' = K_c = |K_c'|$ and that the antiferromagnetic magnetizations at $K < -K_c$ can be obtained from the opposite equal magnetizations at $K > K_c$. In the special case $z = 3$ we get

$$m_\pm = \pm \frac{1}{1 - 2\mathrm{e}^{-|K|}} \sqrt{\frac{\mathrm{e}^{|K|} - 3}{\mathrm{e}^{|K|} + 1}}. \tag{36}$$

For $K > K_c$ equation (36) gives the two possible magnetizations and for $K < K_c' = -K_c$ the magnetization of adjacent shells. An expansion analogue to equations (33)–(35) yields the values $\beta = \frac{1}{2}$, $\gamma = 1$ and $\delta = 3$. These mean-field indices are expected from an infinite-dimensional lattice as the Bethe lattice.

In contrast we encounter for $q > 2$ second-order transitions at the Bethe–Peierls point K_c with percolation indices and at K_c' with mean-field indices. In addition a first-order transition at $K_c'' < K_c$ occurs. Investigating the stability of the system by the free energy, one would obtain only a first-order transition at $K_c'' < K_B < K_c$, where $f_B(x_1, K_B) = f_B(x_2, K_B)$ holds [15]. The state x_3 would be metastable and K_c', K_c would correspond to spinodal points.

5. Logistic equation

The recursion formula (4) can be extrapolated to $q = 1$ without leading to a trivial result. m and w_2^c from (10) and (18) remain nonzero. Due to the normalization factor in (9) and (18) the limit $q \to 1$ is equivalent to taking $d/dq\, A(q)|_{q=1}$ for an observable A. For $K > 0$ [24] this limit corresponds to bond percolation with a probability $p = 1 - \mathrm{e}^{-K}$. At least for the Bethe lattice with $z = 3$ we can interpret the $q \to 1$ limit by another model valid for all K and L, namely the logistic equation. The recursion formula (4) reads in this case as

$$x_n = \mathrm{e}^{-K} - \mathrm{e}^{K} u (x_{n+1})^{z-1} \tag{37}$$

with the parameter

$$u = \mathrm{e}^{-K}(\mathrm{e}^{-K} - 1)\mathrm{e}^{-L}. \tag{38}$$

Performing for $z = 3$ a linear transformation

$$y_n = \tfrac{1}{4}\left(2 + \left(\sqrt{1 + 4u} - 1\right)\mathrm{e}^{K} x_n\right) \tag{39}$$

we obtain for y_n the logistic equation

$$y_n = r y_{n+1}(1 - y_{n+1}) \tag{40}$$

with the control parameter

$$r = 1 + \sqrt{1 + 4u}. \tag{41}$$

Therefore, we found a correspondence between the thermal equilibrium properties of the Pott's model on a ($z = 3$)-Bethe lattice and the logistic equation. The thermal distributions depend on the boundary condition and this dependence is described by the logistic equation. Equation (40) has been studied extensively in the literature [25]. We notice a universal property that the control parameter r depends only on the combination u, but not on K or L separately. Systems at constant u have the same type of solutions. u must satisfy the inequality $u > -\frac{1}{4}$, otherwise (37) has only chaotic solutions, which means x_n for any $n > 1$ is sensitive to the boundary condition x_R. Since for the control parameter in (41) $r > 1$ holds, the trivial fixed point $y = 0$ of (40) is always unstable. In the range $1 < r < 3$ or $-1 < 4u < 3$ corresponding to the range (31) at $L = 0$ we encounter the stable fixed point

$$x = \frac{\mathrm{e}^{-K}}{2u}\left(\sqrt{1 + 4u} - 1\right). \tag{42}$$

From the general formula (18) for the correlation function we obtain w_2^c in the fixed-point case:

$$w_2^c(\lambda, r) = \lambda e^{-L} x^3 \left(\tfrac{1}{2}(1 - \sqrt{1+4u})\right)^r . \tag{43}$$

The signs in (43) for $\lambda = 1$ exhibit for $K < 0$ or $u > 0$ an antiferromagnetic ordering. Increasing the value of u beyond $\frac{3}{4}$ (in the antiferromagnetic region) a series of period doubling occurs, followed by a region of deterministic chaos. Above $u = 2$ only chaotic solutions remain. The solutions corresponding to a $\tau = 2$ orbit read

$$x_\pm = \frac{e^{-K}}{2u}\left(1 \pm \sqrt{4u-3}\right). \tag{44}$$

According to (22) this orbit is stable in the range $3 < 4u < 5$. Note that negative x are not excluded by positivity of $w_1(\sigma)$ in the limit $q \to 1$. For the correlation function we find from (18)

$$w_2^c(\lambda, r) = \lambda(-1)^r \left(x_{\alpha_0} x_{\alpha_r}\right)^{3/2} e^{-L}(1-u)^{r/2} \tag{45}$$

where α_i denote the type of the sites $i = 0, r$. Despite the formal appearance of $\sqrt{1-u}$ the function (45) is analytic at $u = 1$. For $\frac{3}{4} < u < 1$ one observes the same antiferromagnetic ordering as in (43) for $K < 0$. In the range $1 < u < \frac{5}{4}$ we find a period doubling of the sign pattern. The transition point $u = 1$ implies a superstable cycle of the logistic equation. At $u = 1$ all correlation vanish and the system disintegrates into uncorrelated shells of spins with magnetization $m(x_-) = 1$ at odd points and $m(x_+) = 1 - \exp(-L)x_+^3$ at even points. At $L = 0$ the ratio $1/x_+$ of expectation values $\langle \delta_{\sigma,\bar{\sigma}} \rangle$ and $\langle 1 - \delta_{\sigma,\bar{\sigma}} \rangle$ taken on a branch is equal to the golden mean value $1/x_+ = (\sqrt{5}-1)/2$ indicating maximal disorder. In both cases, $\tau = 1, 2$, the decay of the correlations is a function of u only, whereas the amplitudes of w_2^c depends on both K and L.

From the properties of the recursion formula (37) or the equivalent logistic equation (40) we get the phase diagram of the Potts model extrapolated to $q = 1$ in the K, L-plane. This is depicted in figure 4 for the case $z = 3$.

The lines $u = -\frac{1}{4}$ and $u = 2$ separate the chaotic regimes $u > 2$ and $u < -\frac{1}{4}$ from regions with a possible stable orbit. For ferromagnetic coupling ($K > 0$) above $u = -\frac{1}{4}$ we have always the fixed-point solution given by (47) with $m \neq 0$ except the line $L = 0$, where $m = 0$ only for $K < K_c$ holds. The line $L = 0$ touches $u = -\frac{1}{4}$ at $K = K_c = \ln 2$. Therefore, we have a second-order transition which can be identified with the transition of bond percolation. The critical indices $\beta = 1, \gamma = 1$ and $\delta = 2$ we have derived already in the general case. The connection to percolation we postpone to the last section. For antiferromagnetic coupling ($K < 0$) we find the line $u = \frac{3}{4}$ separating the fixed-point solution from the $\tau = 2$ orbit. The line $u = \frac{3}{4}$ intersects the $L = 0$ axis at $K_c' = -\ln\frac{3}{2}$. The transition at K_c' is more complicated, since one has to distinguish average and staggered magnetization $m_{0,s}$ from equation (19). Inserting (44) into (19) we find near K_c'

$$m_0 \sim K_c' - K \tag{46}$$

$$m_s \sim \sqrt{K_c' - K} \tag{47}$$

which implies critical indices $\beta_0 = 1$, resp. $\beta_s = \frac{1}{2}$. By expanding the $\tau = 2$ solution with a staggered field L_s we have $m_0|_{L_s=0,K=K_c'} \sim L^{1/\delta_0}$ with $\delta_0 = 1$, and $m_s|_{L=0,K=K_c'} \sim L_s^{1/\delta_s}$ with $\delta_s = 3$. Likewise we obtain from $\chi_{0,s} = \partial m(x_+)/\partial L_{0,s}\big|_{L_i=0} \sim (K_c' - K)^{-\gamma_{0,s}}$ the indices $\gamma_0 = 0$ and $\gamma_s = 1$ in agreement with the scaling relation $\beta(\delta - 1) = \gamma$. Occurence of two order parameters $m_{0,s}$ at $K = K_c'$ means a crossover phenomenon. Approaching K_c' in the K, L-plane at $L_s = 0$, m_0 is the relevant order parameter with indices β_0, γ_0 and approaching

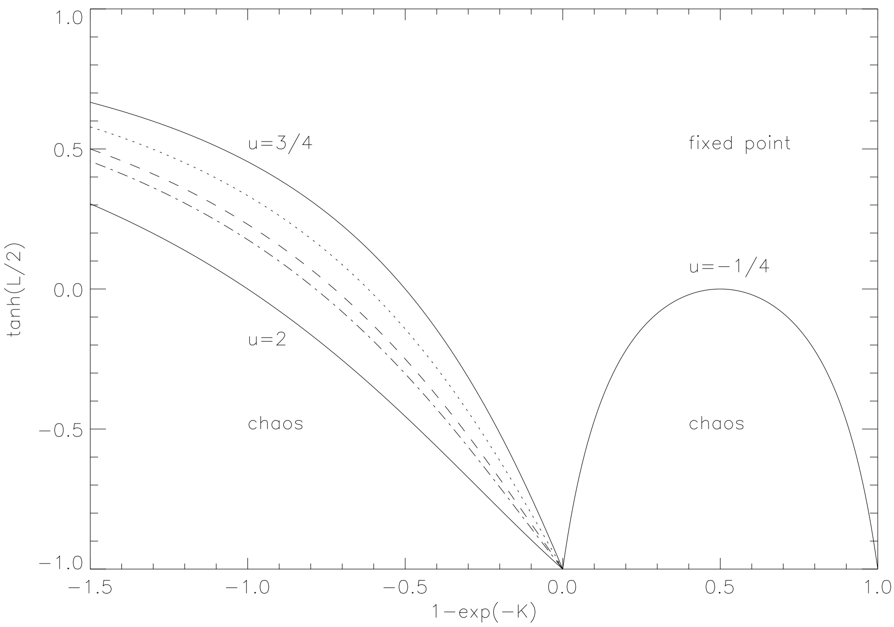


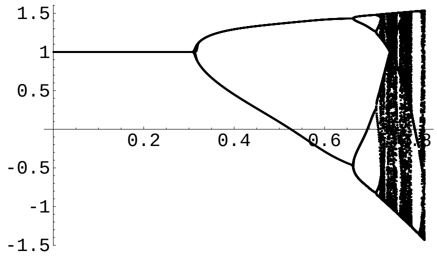
Figure 4. Phase diagram of $q = 1$ Potts model in the plane of $1 - \exp(-K)$ and $\tanh(L/2)$. Above the solid curves $u = -\frac{1}{4}$ and $u = \frac{3}{4}$ we have a phase descibed by a fixed point. Between the latter and the dashed curve $u = \frac{5}{4}$ there exists a period 2 solution with a superstable cycle (dotted curve) at $u = 1$. Period doublings happen between $u = \frac{5}{4}$ and $u = u_\infty = 1.401$ (dashed-dotted curve). The region of deterministic chaos lies between $u = u_\infty$ and $u = 2$. Below $u = 2$ and $u = -\frac{1}{4}$ the observables depend sensitively on their values at the surface (chaos).

K_c' in the K, L_s-plane at $L = 0$, m_s is relevant with indices β_s, γ_s. In contrast to K_c, the point ($L = 0$, K_c') corresponds to a multicritical point. Above $u = \frac{5}{4}$ the $\tau = 2$ orbit is to be replaced by a $\tau = 4$ orbit. Further period doublings occur in the region $u < u_\infty = 1.401\,1551$. In the region of deterministic chaos $u_\infty < u < 2$ stable orbits are surrounded by chaotic solutions. We used $z = 3$ since it allows explicit calculation of w_2 and x. We expect a similar pattern [26] for all odd z, since the Feigenbaum route to chaos depends only on the property of the recursion formula (37): that its right-hand side has a single maximum at $x = 0$. For even z the extremal value $x = 0$ corresponds to a saddle point. We found numerically that for $z = 4, 6, 8$ apart from the period 2 solution no further period doubling occurs. Comparison of the cases $z = 5$ and $z = 4$ is given in figure 5. The rich structure observed for $z = 5$ in x as function of u at $L = 0$ is absent in $z = 4$.

6. Cluster interpretation

In the previous section we learned that the $q = 1$ limit of the Potts model and the logistic equation are connected by the fact that the latter describes the mean-field equation for the equilibrium properties of the Potts model. Conventionally [9] the $q = 1$ ferromagnetic Potts model is interpreted as bond percolation. In this section we wish to investigate to what extent the more interesting antiferromagnetic case ($K < 0$) and percolation are related.

Any observable in the Potts model can be obtained from the two-point correlation w_2^c. The bridge to percolation is the interpretation of w_2^c from equations (43) and (45) as the probability



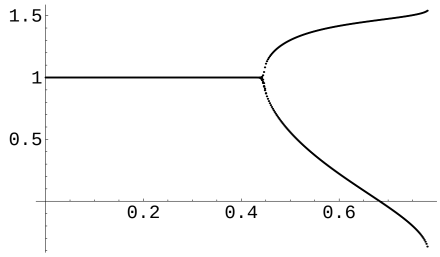


Figure 5. Fixed points and orbits as function of $\exp(-K)(\exp(-K)-1)$ for $z = 5$ (left) and $z = 4$ (right).

that the origin and a point at distance r are in the same cluster. $w_2^c \geqslant 0$ can be achieved for $r > 0$ by selecting $\lambda(r)$. We have to find a more general percolation as the usual bond percolation, since $w_2^c(\lambda, r)$ depends on two parameters (L and u) and distinguishes between odd and even origins α_0. In addition, $w_2^c(\lambda, r)$ may violate $0 \leqslant w_2^c(\lambda, 0) \leqslant 1$. As observed by Leads [27] bond percolation is equivalent to a cluster-growth model, where from a starting point further links are added with probability p and rejected with probability $1-p$. Today this algorithm is known as the Wolff algorithm [28]. $w_2^c(\lambda, 1) \leqslant 1$ in all cases suggests that we should start in a generalized growth model with a link instead of a site. This link is chosen with probability p_L and enlarged to a cluster by the following algorithm. Each site of a link will be continued to $i-1$ further links with probability p_i with $i = 1, \ldots, z$. If we label the sites of an AB-lattice as the Bethe lattice with $\alpha = \pm 1$ corresponding to a possible antiferromagnetic order, these probabilities $p_{i,\alpha}$ may depend on the type α of the site. Continuing this procedure we construct a cluster, which can be characterized by $E_{i,\alpha}$ equal to the number of sites of type α connected to i neighbouring sites. Using the geometry of a Bethe lattice $E_{1,\alpha}$ can be expressed by the other. In the case $z = 3$ there are two relations

$$E_{1,\alpha} = 2E_{3,-\alpha} - E_{3,\alpha} + E_{2,-\alpha} - E_{2,\alpha} + 1. \tag{48}$$

Figure 6 gives an example for $z = 3$ with $E_{2-} = 2$, $E_{3+} = 1$ and $E_{3-} = E_{2+} = 0$. It occurs with probability $p_L p_{1+}^2 p_{1-} p_{2-}^2 p_{3+}$. A general cluster with numbers $E_{i,\alpha}$ may begin on either site with type α_0 of the first link. Its probability under the condition of presence of the first link is given by

$$w_{\alpha_0}(E|L) = g(E) \prod_{i,\alpha} (p_{i,\alpha})^{E_{i,\alpha}} \tag{49}$$

where $g(E)$ denotes the combinatorial number of different clusters with given number $E_{i,\alpha}$. The probability w_{α_0} for any cluster

$$w_{\alpha_0} = \sum_E w_{\alpha_0}(E|L) \tag{50}$$

needs not be 1. In analogy to the Bolzmann distribution for a tree as in section 2, the probability w_{α_0} satisfies a recursion formula. For a growth model with period 2 we get

$$w_{-\alpha_0} = p_{1,\alpha_0} + 2p_{2,\alpha_0} w_{\alpha_0} + p_{3,\alpha_0} w_{\alpha_0}^2. \tag{51}$$

Solving the two equations (51) for $w_{\alpha_0}(p)$ and expanding w_{α_0} in powers of $p_{i,\alpha}$ the combinatorical factor $g(E)$ in equation (49) can be determined. The explicit form of $w_\alpha(p)$ is not needed if we are interested only in the correlation function, which is the probability that one site (origin) at the starting link appearing with probability p_L is connected (see figure 6)

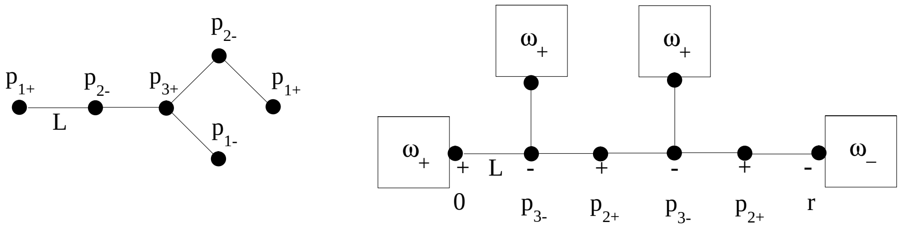


Figure 6. The left part gives an example for a cluster growing from the link L. The right part shows the graph for correlations between an origin 0 and a point in distance $r = 5$. Squares denote any possible contuation at $E_{3\pm}$ points occurring with probability $w_\pm$.

by $r - 1$ further links to a site in distance r. The probability ρ_α for such links connecting a site of type $-\alpha$ with a neighbour of type α is given by

$$\rho_\alpha = p_{2,\alpha} + p_{3,\alpha} w_\alpha. \tag{52}$$

Multiplying all probabilities we find the correlation function connecting 0 with a point at distance r, as in figure 6,

$$\Gamma_\alpha(r) = w_\alpha \cdot p_L \cdot \begin{cases} (\rho_+\rho_-)^{\frac{r-2}{2}} w_\alpha \rho_\alpha & r > 0 \quad \text{even} \\ (\rho_+\rho_-)^{\frac{r-1}{2}} w_{-\alpha} & r \text{ odd}. \end{cases} \tag{53}$$

Note that $\Gamma_\alpha(0)$ is neither defined nor needed. We can consider q_α, w_α as independent parameters and do not need to perform the elimination of equations (51), (52). In the case of α-independent probabilities $p_{i,\alpha} = p_i$, equations (51) and (52) can be solved leading to

$$\rho = \tfrac{1}{2}\left(1 - \sqrt{(1-2p_2)^2 - 4p_1p_3}\right) \tag{54}$$

$$w = \frac{1}{p_3}(\rho - \rho_2). \tag{55}$$

In this case (53) can be extrapolated to $r = 0$ by introducing p_0 as

$$p_0 = \frac{1}{\rho} w_L^p \tag{56}$$

being the probability for the presence of a single point which leads to the simple formula

$$\Gamma(r) = p_0 \cdot \rho^r. \tag{57}$$

Due to the constraint (48) the relation between $p_{i,\alpha}$ and q_α, w_α is not unique. In addition, the link probability p_L in equation (53) or p_0 in equation (57) is a free parameter. A model with greatly reduced freedom of $p_{i,\alpha}$ is the α-independent bond percolation, where the growth parameters p_i are given in terms of a link probability p_L through

$$p_1 = (1-p_L)^2 \qquad p_3 = p_L^2 \qquad p_2^2 = p_1 p_3. \tag{58}$$

Inserting equation (58) into equations (54)–(56) we find for $p_L \leqslant \frac{1}{2}$

$$\rho = p_L \qquad w = p_0 = 1 \tag{59}$$

and for $p_L \geqslant \frac{1}{2}$

$$\rho = 1 - p_L \qquad w = \left(\frac{1-p_L}{p_L}\right)^2 \qquad p_0 = \left(\frac{1-p_L}{p_L}\right)^3. \tag{60}$$

This shows the percolation phase transition at $p_L = \frac{1}{2}$. In the growth model we avoid the notion of an 'infinite' cluster for $p_L \geqslant \frac{1}{2}$ by the probability $p_0 < 1$, that a point belongs to a cluster at all. Comparing equation (53) with the correlation function $w_2^c(\alpha, r)$ of the Potts model equation (45) in the fixed-point regime we can identify

$$p_0 = x^3 \mathrm{e}^{-L} = 1 - m \tag{61}$$

$$\rho = |\epsilon| = \tfrac{1}{2}\left|\sqrt{4u+1} - 1\right|. \tag{62}$$

Both, percolation and the Potts model have in common that the decay parameter ϵ (resp. ρ) of the correlation function and the magnetization $1 - m$ (resp. the point probability p_0) can be chosen independently. At $L = 0$ we have in both cases only one free parameter. Using the bond percolation parametrization (58) for p_i and expressing u in terms of e^{-K} we find the link probability

$$p_L = |\mathrm{e}^{-K} - 1| \tag{63}$$

$$p_0 = x^3 = 1 - m = \begin{cases} 1 & K_c' < K < K_c \\ (\mathrm{e}^K - 1)^{-3} & K > K_c. \end{cases} \tag{64}$$

The critical points $p_L = \frac{1}{2}$ correspond to $K = K_c, K_c'$. Whereas for ferromagnetic coupling $(K > 0)$ K_c is inside the validity of the fixed-point regime, the antiferromagnetic transition K_c' is located at the border. This is due to the change of a fixed-point solution into an orbit $\tau = 2$ solution. For $K < K_c'$ we have to compare the general formula (45) for $w_2^c(\alpha, r)$ with Γ^0 from the percolation model (53). Identification of the decay parameter leads to

$$\rho_\alpha = \epsilon_\alpha = \tfrac{1}{2}\left|1 + \alpha\sqrt{4u - 3}\right|. \tag{65}$$

Since the amplitude involves the link probability p_L which can be no longer eliminated with an argument leading to (56), we can compare only the ratios

$$\frac{w_\alpha}{w_{-\alpha}} = \frac{\epsilon_\alpha}{\epsilon_{-\alpha}} = \left|\frac{x_\alpha}{x_{-\alpha}}\right|. \tag{66}$$

From the value of $w_2^c(\alpha, 1)$ we find

$$w_+ w_- p_L = (1 - u)^2 \left(\frac{\mathrm{e}^{-K}}{u}\right)^3. \tag{67}$$

Any cluster-growth model with (65) for the decay parameter and (66), (67) for w_α and p_L will have the same correlation function as the $q = 1$ Potts model with $z = 3$ in the period 2 phase. In particular, the ratio $|x_-|/x_+$ of the period 2 solution of the logistic equation appears to be the ratio w_-/w_+ of having a cluster starting at a point of type $\alpha = \pm 1$.

7. Conclusions

The thermodynamic properties of the q-state Potts model on a Bethe lattice can be exactly calculated by mean-field methods. The parameters x_n and a_n (magnetization and partition sum of a branch) can be determined recursively from the values x_R, a_R at the surface. The formula for x_n relevant for local correlations expressed by w_2 may have a stable fixed point or orbit. This means that w_2 is insensitive to the boundary conditions for sufficiently large distances from the surface. In contrast, a_n relevant for global quantities as the free energy and its derivatives is always sensitive to x_R and a_R. This reflects the difficulty to obtain the Bethe lattice by extrapolating Cayley trees to large sizes, since the influence of the surface points and the transition region (if x_R is not exactly at the fixed point) cannot be neglected. A natural way to treat the Bethe lattice is to consider a sublattice of n_0 shells, where the distance R–n_0 is large enough that x_n can be replaced by a fixed point. Generalizing the method of

Baxter [5] to any q a free energy for this subsystem can be found [23] from the magnetization and the energy/bound inside n_0. However, due to surface instabilities this function should not be used to reject fixed points on the basis of its value. Instead one should use the stability of the fixed point or the orbit. Applying this criterion to the zero-external-field case we find for $q > 2$ and ferromagnetic coupling a first-order transition and a second-order transition at the Bethe–Peierls temperature with critical indices of the percolation class. The transition to antiferromagnetic ordering at negative coupling is also of second order, but with indices of the mean-field class. The Ising case $q = 2$ is exceptional, since its ferromagnetic transition at K_c and its antiferromagnetic transition at $K'_c = -K_c$ are related and are both of second order with mean-field indices. Of particular interest is the extrapolation to $q = 1$. In the case of $z = 3$ the recursion relation for x_n is identical to the logistic equation. This equivalence holds for any K and L, whereas the usually discussed equivalence with bond percolation is only valid for $K > 0$ and $L = 0$. For antiferromagnetic coupling $K < 0$ we encounter in the phase diagram the rich structure of the logistic equation (sequence of period doubling, supercycles, deterministic chaos). We found that the first period doubling at $L = 0$ of the logistic equation corresponds in the $q = 1$ Potts model to a multicritical point K'_c, where two second-order transitions exhibit a crossover: one with critical indices of the mean-field class and a second with indices $\delta = \beta = 1$ and $\gamma = 0$. The first superstable cycle of the logistic equation corresponds to a situation where uncorrelated spins are antiferromagnetically ordered in shells. One shell has magnetization $m_- = 1$ and the neighbouring shell a value m_+ related to the golden mean. By a Feigenbaum-type argument a similar pattern should arise for any odd z. On the Bethe lattice one can generalize bond percolation to a cluster-growth model, which can be interpreted by the ($q = 1$)-Potts model also for $L \neq 0$ and $K < K'_c$. Since the latter is also related to the logistic equation, the cluster-growth model may serve as a dynamical model for applications of the logistic equation in economical problems [29].

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